

ACCURACY OF SEVERAL MULTIDIMENSIONAL CRYSTAL-REFINABLE FUNCTIONS

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ABSTRACT. Let Γ be a crystal group in \mathbb{R}^d . A function $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is said to be Γ -refinable if it is a linear combinations of finitely many of the rescaled and translated functions $f(\gamma^{-1}(Ax))$, where the *translations* γ are taken on a crystal group Γ , and A is an expansive dilation matrix such that $A\Gamma A^{-1} \subset \Gamma$. Γ -refinable functions satisfy a refinement equation $f(x) = \sum_{\gamma \in \Gamma} d_\gamma f(\gamma^{-1}(Ax))$ with d_γ being $r \times r$ matrices, and $f(x) = (f_1(x), \dots, f_r(x))^T$. The accuracy of f is the highest degree p such that all multivariate polynomials $q(x)$ of degree $\deg(q) < p$ are exactly reproduced from linear combinations of translates of f along the crystal group Γ . In this paper, we determine the accuracy p from the matrices d_γ . Moreover, we deduce from our conditions, a characterization of accuracy for some lattice refinable vector function F , which simplifies the classical conditions.

1. INTRODUCTION

Crystal groups (Crystallographic groups or space groups) Γ , are groups of isometries of \mathbb{R}^d that generalize the notion of translations on a lattice, allowing to move using different (rigid) movements in \mathbb{R}^d following a bounded pattern that is repeated until it fills up space.

Precisely (see [7]):

Definition 1.1. A *crystal group* is a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{R}^d)$ such that $\text{Isom}(\mathbb{R}^d)/\Gamma$ is compact, where $\text{Isom}(\mathbb{R}^d)$ is endowed with the topology of pointwise convergence.

Or equivalently, one can define a *crystal group* to be a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{R}^d)$ such that there exists a compact *fundamental domain* P for Γ .

2010 *Mathematics Subject Classification.* Primary 42C40, Secondary 52C22.

Key words and phrases. Crystallographic groups, Accuracy, Refinement equation, Composite dilations.

Intuitively, a crystal should have a bounded pattern that is repeated until it fills up space, i.e. there exists a bounded closed set P such that

$$\bigcup_{\gamma \in \Gamma} \gamma(P) = \mathbb{R}^d \text{ and } \gamma(P^\circ) \cap \gamma'(P^\circ) \neq \emptyset \text{ then } \gamma = \gamma',$$

where P° is the interior of P .

This set is called *fundamental domain*, which corresponds to the fundamental domain for lattices, only that here its shape can be much more general.

Note that a particular case of crystal group is the group of translations on a lattice.

In this paper we are interested in systems of functions, generated by dilations and movements along a crystal group.

Let us start recalling the necessary definitions.

Definition 1.2. Let Γ be a crystal group. We will say that A is a Γ -admissible matrix, if A is an expanding affine map and $A\Gamma A^{-1} \subset \Gamma$.

It is easy to see that if A is a Γ -admissible matrix, then $m = |\det A|$ is an integer. Therefore, the quotient group $\Gamma/A\Gamma A^{-1}$ is of order m .

A function $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is Γ -refinable with respect to A and Γ if it is a linear combinations of the rescaled and ‘translated’ functions $f(\gamma^{-1}Ax)$, where the ‘translates’ $\gamma \in \Gamma$ are movements on Γ . Precisely, $f(x) = (f_1(x), \dots, f_r(x))^T$ satisfies a *refinement equation*, *dilation equation* or *two-scale difference equation* of the form

$$(1) \quad f(x) = \sum_{\gamma \in \Gamma'} d_\gamma f(\gamma^{-1}Ax)$$

for some finite $\Gamma' \subset \Gamma$ and some $r \times r$ matrices d_γ . These matrices are called *coefficients of the refinement equation*.

Refinable functions with respect to A and Γ are related to *Crystal Wavelets* and *Wavelets with composite dilations* [8], [9], [14].

In this paper we study the general multidimensional, multifunction case ($d \geq 1, r \geq 1$) with a Γ -admissible matrix A . We seek to determine one fundamental property of the space spanned by a Γ -refinable function f based on the coefficients d_γ . That property is the *accuracy* of f :

Definition 1.3. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$, the *accuracy* of f is the largest integer p such that *all* multivariate polynomials $q(x) = q(x_1, \dots, x_d)$ of $\deg(q) < p$ lie in the space

$$(2) \quad S(f) = \overline{\text{span}} \left\{ \sum_{i=1}^k d_{\gamma_i} f(\gamma_i x) : d_{\gamma_i} \in \mathbb{C}^{1 \times r} \right\},$$

which is the closure of all finite linear combinations of Γ -translates of f . As usual, equality of functions is interpreted as holding almost everywhere (a.e.). Note that in fact, the *accuracy* is a property of the space $S(f)$, but since the space is generated by Γ -translates of the function f , we will talk in-distinctively about the accuracy of f , or $S(f)$.

Accuracy has played an important role in both approximation theory and in wavelet theory. In approximation theory, it is closely related to the approximation properties of shift invariant spaces. In wavelet theory, one of the most successful and systematic ways of constructing smooth, compactly supported, orthonormal wavelet bases for $L^2(\mathbb{R})$ is based on the factorization of a symbol which determines a scaling function [6]. This factorization of the symbol is closely related to the accuracy of the scaling function. If the scaling function has accuracy p , then the corresponding wavelet will have p zero moments. Accuracy is necessary for a refinable function to be smooth, although it is not sufficient. General results of accuracy can be found in [3, 4, 5, 11] and references therein.

Our goal in this paper is to obtain necessary and/or sufficient conditions for a vectorial crystal refinable function f to have accuracy p . In this direction, our first result establishes necessary conditions for $S(f)$ (2) with f an arbitrary function, to have accuracy p . For the case that f is a Γ -refinable function, we will give necessary and sufficient conditions to ensure that f has accuracy p . In the case in which f satisfies a refinement equation (1) with $f : \mathbb{R}^d \rightarrow \mathbb{C}$, we can associate a vector function (f_1, \dots, f_r) on \mathbb{R}^d , that satisfies a refinement equation (using translations on a lattice). Using our approach, accuracy conditions on the coefficients of the refinement equation are much simpler than in the general case (see Theorem 4.8).

Following the ideas in [3] we consider together the monomials $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ of a given degree in a single array. With this key observation, it is surprisingly simpler than expected to move from one to several variables.

2. CRYSTAL GROUPS

Definition 2.1. A *crystal group* is a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{R}^d)$ such that $\text{Isom}(\mathbb{R}^d)/\Gamma$ is compact, where $\text{Isom}(\mathbb{R}^d)$ is endowed with the topology of pointwise convergence.

Or equivalently, one can define a *crystal group* to be a discrete subgroup $\Gamma \subset \text{Isom}(\mathbb{R}^d)$ such that there exists a compact *fundamental domain* P for Γ .

The theorem of Bieberbach [1] yields the following:

Theorem 2.2 (Bieberbach). *Let γ be a crystal subgroup of $\text{Isom}(\mathbb{R}^d)$. Then*

- (1) $\Lambda = \Gamma \cap \text{Trans}(\mathbb{R}^d)$ is a finitely generated abelian group of rank d which spans $\text{Trans}(\mathbb{R}^d)$, and
- (2) the linear parts of the symmetries $\text{ad}(\Gamma) \cong \Gamma/\Lambda$, the point group of Γ , is finite.

(See also [12], IV-4). Here $\text{Trans}(\mathbb{R}^d)$ stands for translations of \mathbb{R}^d .

We will denote the point group of Γ by G . and call (Γ, G, Λ) a crystal triple.

Remark 2.3.

- Note that the set Λ is not empty by Bieberbach's theorem [1]. Moreover, Λ consists of translations on a lattice \mathcal{L} which is isomorphic to \mathbb{Z}^d .

We will denote by L and L^* the fundamental domains of the lattices \mathcal{L} and its dual, \mathcal{L}^* respectively. Here $\mathcal{L} = R(\mathbb{Z}^d)$ with R an invertible $d \times d$ matrix and hence $\mathcal{L}^* = (R^*)^{-1}(\mathbb{Z}^d)$.

- The Point Group G of Γ is a finite subgroup of $\mathbf{O}(d)$, the orthogonal group of \mathbb{R}^d , that preserves the lattice of translations, i.e. $G\mathcal{L} = \mathcal{L}$.

General results on crystal groups, can be found for example in [10], [17], [13], [1], [2].

Note that the simplest example of a crystal group is the group of translations on a lattice \mathcal{L} , i.e. $\Gamma = \{\tau_k : k \in \mathcal{L}\}$, where $\tau_k(x) = x + k$.

One very important class of crystal groups, are the *splitting crystal groups*:

Definition 2.4. Γ is called a *splitting crystal group* if it is the semidirect product of the subgroups Λ and G . In this case $\Gamma = G \ltimes \Lambda$ and for each $\gamma, \tilde{\gamma} \in \Gamma$, with $\gamma = (g_i, \tau_k)$ and $\tilde{\gamma} = (g_j, \tau_l)$, we have $\tilde{\gamma} \cdot \gamma = (g_j g_i, \tau_{k g_i^{-1}(l)})$ where $g_i, g_j \in G$, $\tau_k, \tau_l \in \Lambda$ and $\gamma(x) = g_i(x + k)$.

Every crystal group is naturally embedded in a splitting group, and very often arguments for general groups can be relatively easily reduced to the splitting case and then be proved for that simpler case. This justifies, that from now on we will only consider splitting crystal groups.

For simplicity of notation, for each $\gamma \in \Gamma$ we will use the notation $\gamma = (g_i, k)$ instead of (g_i, τ_k) . If $\gamma = (g_i, k)$ and $\tilde{\gamma} = (g_j, l)$, then $\tilde{\gamma} \cdot \gamma = (g_j g_i, k + g_i^{-1}(l))$.

Example 2.5. Consider the vectors $u = (0, 1)$ and $v = (1, 0)$ and let S be the symmetry with respect to the X -axis (i.e $S(x, y) = (x, -y)$).

Let Γ be the group generated by $\{\tau_u, \tau_v, S\}$. Then $\Lambda = \{\tau_l : l \in \mathcal{L}\}$ where $\mathcal{L} = \{\alpha u + \beta v : \alpha, \beta \in \mathbb{Z}\}$ and $G = \{Id, S\}$. The fundamental domain P is the rectangle of vertices $\{(0, 0); (1, 0); (0, 1/2); (1, 1/2)\}$

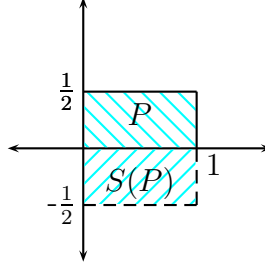


Figure 1. $P \cup S(P)$ is the fundamental domain for Λ .

3. NOTATION

The notation of this paper is complicated due to the dimension and the multiplicity of functions.

We use the standard multi-index notation $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, where $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d)$ with each α_i a nonnegative integer. The degree of α is $|\alpha| = \alpha_1 + \dots + \alpha_d$. The number of multi-indices α of degree s is $d_s = \binom{s+d-1}{d-1}$. We write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $i = 1, \dots, d$.

For each integer $s \geq 0$ we define the vector-valued function $X_{[s]} : \mathbb{R}^d \rightarrow \mathbb{C}^{d_s}$ by

$$X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}, ; x \in \mathbb{R}^d.$$

For our purposes we need define two special matrices, $A_{[s]}$ and $Q_{[s,t]}$ for integers $s, t \geq 0$. Given a matrix A , we define the matrices $A_{[s]}$ and $Q_{[s,t]}$ by

$$\begin{aligned} X_{[s]}(Ax) &= A_{[s]}X_{[s]}(x), \\ X_{[s]}(x-y) &= \sum_{t=0}^s Q_{[s,t]}(y)X_{[t]}. \end{aligned}$$

These matrices have three properties that will be of great importance.

Lemma 3.1. *Let $A \in \mathbb{R}^{d \times d}$ be a matrix, and \mathcal{L} be the lattice associated to the crystal group Γ (see Remark 2.3). Then:*

- (1) *If A is an expansive matrix then $A_{[s]}$ is an expansive matrix for each $s \geq 0$.*
- (2) *If A is an invertible matrix then $Q_{[s,t]}(Az) = A_{[s]}Q_{[s,t]}(z)A_{[t]}^{-1}$.*

- (3) Let $B_t \in \mathbb{C}^{d_t \times r}$ be given matrices, for $0 \leq t \leq s$. If $\sum_{t=0}^s Q_{[s,t]}(Al)B_t = 0$ for each $l \in \mathcal{L}$, then $B_t = 0$ for $0 \leq t \leq s$.

The proof of the previous lemma as well as the explicit form and properties of these matrices can be seen in [3].

From the matrices $A_{[s]}$ and $Q_{[s,t]}$ we obtain the following definition.

Definition 3.2. Let (Γ, G, Λ) be a splitting crystal triple. Let $\gamma \in \Gamma$, $\gamma = (b, l)$ then we define the matrices $\tilde{Q}_{[s,t]}$ by

$$\tilde{Q}_{[s,t]}(\gamma) = Q_{[s,t]}(l)b_{[t]}^{-1}.$$

Lemma 3.3. Let (Γ, G, Λ) be a splitting crystal triple, and A an invertible matrix such that $A\Gamma A^{-1} \subset \Gamma$. We then have:

- (1) $\mathcal{Q}_{[s,s]}(\tau_l) = Id$.
- (2) $\mathcal{Q}_{[s,0]}(\gamma) = (-1)^s X_{[s]}(l)$ for each $\gamma = (b, l) \in \Gamma$.
- (3) $\mathcal{Q}_{[s,t]}(\gamma_1 \gamma_2) = \sum_{u=t}^s \mathcal{Q}_{[s,u]}(\gamma_2) \mathcal{Q}_{[u,t]}(\gamma_1)$.
- (4) $\mathcal{Q}_{[s,t]}(A\gamma A^{-1}) = A_{[s]} \mathcal{Q}_{[s,t]}(\gamma) A_{[t]}^{-1}$.

The proof the previous lemma is immediate from Lemma 3.1 and Lemma 4.1 of [3].

Given a collection

$$\{v_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,r}) \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\},$$

of row vectors of length r , we shall associate special matrices and functions, which play an important role in our analysis of accuracy. We use the following notation throughout the paper.

We group the v_α by degree to form $d_s \times 1$ column vectors $v_{[s]}$ with block entries that are the $1 \times r$ row vectors v_α . Specifically, we set

$$v_{[s]} = [v_\alpha]_{|\alpha|=s}, \quad 0 \leq s < p.$$

Note that, when $\alpha = 0$ then $v_{[0]} = [v_0] = v_0$.

$$(3) \quad y_{[s]}(\gamma) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma) v_{[t]},$$

where $\gamma = (b, l)$ and $b_{[t]}$ is the matrix that satisfies $X_{[t]}(bx) = b_{[t]} X_{[t]}(x)$.

Finally, we define the infinite row vector

$$(4) \quad Y_{[s]} = (y_{[s]}(\gamma))_{\gamma \in \Gamma}.$$

The functions $y_{[s]}$ have the following properties.

Lemma 3.4. *Let $\{v_\alpha \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$ be a collection of vectors and let $y_{[s]}$ be the functions $y_{[s]}(\gamma) = \sum_{t=0}^s \tilde{Q}_{[s,t]}(\gamma)v_{[t]}$. Let γ_1 and γ_2 in Γ , then*

$$y_{[s]}(\gamma_1 \gamma_2) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_2) y_{[t]}(\gamma_1).$$

If $\gamma_2 = (Id, l_2) = \tau_{l_2}$, then the previous equality yields

$$y_{[s]}(\gamma_1 \tau_{l_2}) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(l_2) y_{[t]}(\gamma_1).$$

Proof. For the proof we use Lemmas 4.1, 4.2 and 4.3 of [3]. By definition

$$\begin{aligned} y_{[s]}(\gamma_1 \gamma_2) &= \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_1 \gamma_2) v_{[t]} = \sum_{t=0}^s \sum_{u=t}^s \mathcal{Q}_{[s,u]}(\gamma_2) \mathcal{Q}_{[u,t]}(\gamma_1) v_{[t]} \\ &= \sum_{u=0}^s \mathcal{Q}_{[s,u]}(\gamma_2) \sum_{t=0}^u \mathcal{Q}_{[u,t]}(\gamma_1) v_{[t]} = \sum_{u=0}^s \mathcal{Q}_{[s,t]}(\gamma_2) y_{[u]}(\gamma_1). \end{aligned}$$

□

We will say that translates of f along Γ are Γ -independent if for every choice of row vectors $b_\gamma \in \mathbb{C}^{1 \times r}$,

$$\sum_{\gamma \in \Gamma} b_\gamma f(\gamma x) = 0 \text{ if, and only if, } b_\gamma = 0 \text{ for every } \gamma.$$

Equivalently, for every choice of an infinite row vector $b = (b_\gamma)_{\gamma \in \Gamma}$ with block entries $b_\gamma \in \mathbb{C}^{1 \times r}$,

$$bF(x) = 0 \text{ if, and only if, } b = 0.$$

Here $F(x)$ is the infinite column vector with block entries $f(\gamma(x))$, i.e.

$$(5) \quad F(x) = [f(\gamma(x))]_{\gamma \in \Gamma}.$$

4. CHARACTERIZATION OF ACCURACY

4.1. Necessary conditions for arbitrary functions. In this section, we will present necessary conditions for an arbitrary (not necessarily Γ -refinable) function $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ with Γ -independent translates, to have accuracy p .

Theorem 4.1. *Assume that $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is compactly supported, and that translates of f are Γ -independent. If f has accuracy p then there exists a collection*

$$\{v_\alpha \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$$

of row vectors such that

i): $v_0 \neq 0$.

ii): $X_{[s]}(x) = \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) f(\gamma(x)) = Y_{[s]} F(x)$ for $0 \leq s < p$.

Where $Y_{[s]} = (y_{[s]}(\gamma))_{\gamma \in \Gamma}$ as in (3) and (4).

Proof. Since f has accuracy p , there exist row vectors $w_{\alpha, \gamma} \in \mathbb{C}^{1 \times r}$ such that every polynomial x^α of degree α , $0 \leq |\alpha| < p$ can be written as a finite linear combination of Γ -translates of f .

$$x^\alpha = \sum_{\gamma \in \Gamma} w_{\alpha, \gamma} f(\gamma(x)) \text{ a.e.}$$

For each $\gamma \in \Gamma$, group the vectors $w_{\alpha, \gamma}$ by degree to form the column vectors

$$w_{[s]}(\gamma) = [w_{\alpha, \gamma}]_{|\alpha|=s}.$$

For each $\sigma \in \Gamma$ define the infinite row vector

$$W_{[s]}(\sigma) = (w_{[s]}(\gamma\sigma))_{\gamma \in \Gamma}.$$

Next, set $v_\alpha = w_{\alpha, I}$ (where I is the identity of Γ) and define the vectors $v_{[s]}$ and the matrices of polynomials $y_{[s]}$ by

$$v_{[s]} = [v_\alpha]_{|\alpha|=s} \text{ and } y_{[s]}(\gamma) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma) v_{[t]}.$$

Then, grouping the polynomials x^α by degree, we have for $0 \leq s < p$, that

$$\begin{aligned} X_{[s]}(x) &= [x^\alpha]_{|\alpha|=s} = \left[\sum_{\gamma \in \Gamma} w_{\alpha, \gamma} f(\gamma(x)) \right]_{|\alpha|=s} \\ &= \sum_{\gamma \in \Gamma} w_{[s]}(\gamma) f(\gamma(x)) = W_{[s]}(I) F(x). \end{aligned}$$

Therefore, for each $\sigma = (g, h) \in \Gamma$

$$\begin{aligned} W_{[s]}(\sigma) F(x) &= W_{[s]}(I) F(\sigma^{-1}(x)) = X_{[s]}(g^{-1}(x) - h) \\ &= \sum_{t=0}^s Q_{[s,t]}(h) g_{[t]}^{-1} X_{[t]}(x) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) X_{[t]}(x) \\ &= \left(\sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) W_{[t]}(I) \right) F(x). \end{aligned}$$

Considering our assumption that translates of f are Γ -independent, this implies that

$$W_{[s]}(\sigma) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) W_{[t]}(I),$$

and therefore for each $\gamma \in \Gamma$ we have that

$$(w_{[s]}(\gamma\sigma))_{\gamma \in \Gamma} = \left(\sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) w_{[t]}(\gamma) \right)_{\gamma \in \Gamma}.$$

In particular, taking $\gamma = I$ we obtain

$$w_{[s]}(\sigma) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) w_{[t]}(I) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\sigma) v_{[t]} = y_{[s]}(\sigma).$$

Thus

$$X_{[s]}(x) = \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) f(\gamma(x)) = Y_{[s]} F(x).$$

Consider now the case $s = 0$. Since $y_{[0]}(\gamma) = v_0$ for every $\gamma \in \Gamma$ we have

$$1 = x^0 = X_{[0]}(x) = \sum_{\gamma \in \Gamma} y_{[0]}(\gamma) f(\gamma(x)) = v_0 \sum_{\gamma \in \Gamma} f(\gamma(x)).$$

Therefore $v_0 \neq 0$. □

4.2. Accuracy for Γ -refinable functions. In this section we will obtain necessary and/or sufficient conditions for a Γ -refinable function to have accuracy p .

First, we rewrite equation (1) in matrix form.

Let (Γ, G, Λ) be a splitting crystal triple and A a Γ -admissible matrix. Remember that a function f is Γ -refinable if satisfies

$$f(x) = \sum_{\gamma \in \Gamma} d_{\gamma} f(\gamma^{-1} A x).$$

We consider $F(x)$ the infinite column vector defined by equation (5), i.e. $F(x) = [f(\gamma(x))]_{\gamma \in \Gamma}$. Note that for a given x , only finitely many entries $f(\gamma(x))$ of $F(x)$ are non zero since f has compact support.

Lemma 4.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{C}^d$, $A \in \mathbb{C}^{r \times r}$ a Γ -admissible matrix and F the function defined by $F(x) = [f(\gamma(x))]_{\gamma \in \Gamma}$ (see (5)). Then, the function f is Γ -refinable if and only if $LF(AX) = F(X)$ a.e., where L is the $\Gamma \times \Gamma$ matrix given by $L = [d_{A\gamma A^{-1}\sigma^{-1}}]_{\gamma, \sigma \in \Gamma}$, with $r \times r$ block entries $d_{A\gamma A^{-1}\sigma^{-1}}$.*

Proof. Suppose that f satisfies the Γ -refinement equation (1). We consider $L = [d_{A\gamma A^{-1}\sigma^{-1}}]_{\gamma, \sigma \in \Gamma}$ and we know that

$$F(x) = [f(\gamma(x))]_{\gamma \in \Gamma} = \left[\sum_{\sigma \in \Gamma} d_{\gamma} f(\sigma^{-1}(A\gamma x)) \right]_{\gamma \in \Gamma} \quad \text{by (1).}$$

Hence

$$\begin{aligned} LF(Ax) &= [d_{A\gamma A^{-1}\sigma^{-1}}]_{\gamma, \sigma \in \Gamma} [f(\sigma(Ax))]_{\sigma \in \Gamma} = \left[\sum_{\sigma \in \Gamma} d_{A\gamma A^{-1}\sigma^{-1}} f(\sigma(Ax)) \right]_{\gamma \in \Gamma} \\ &= \left[\sum_{\alpha \in \Gamma} d_{\alpha} f(\alpha^{-1} A\gamma A^{-1} Ax) \right]_{\gamma \in \Gamma} = [f(\gamma(x))]_{\gamma \in \Gamma} = F(x). \end{aligned}$$

Conversely, let us suppose that $LF(Ax) = F(x)$. Then for each $\gamma \in \Gamma$ we have

$$\sum_{\sigma \in \Gamma} d_{A\gamma A^{-1}\sigma^{-1}} f(\sigma(Ax)) = f(\gamma(x)).$$

In particular, if we consider $\gamma = Id$, then

$$\sum_{\sigma \in \Gamma} d_{\sigma^{-1}} f(\sigma(Ax)) = \sum_{\sigma \in \Gamma} d_{\sigma} f(\sigma^{-1}(Ax)) = f(x),$$

therefore f is a Γ -refinable function. \square

The following result characterizes the accuracy of Γ -refinable functions.

Theorem 4.3. *Assume that $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is integrable, compactly supported and satisfies the refinement equation (1). Consider the following statements*

- I) f has accuracy p .
- II) *There exist a collection of row vectors $\{v_{\alpha} \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$ such that*
 - (i) $v_0 \hat{f}(0) \neq 0$ and
 - (ii) $Y_{[s]} = A_{[s]} Y_{[s]} L$ for $0 \leq s < p$ where $Y_{[s]} = (y_{[s]}(\gamma))_{\gamma \in \Gamma}$ as in (3) and (4).

Then we have the following:

- a) *If the translates of f along Γ are independent, then (I) implies (II)*
- b) *(II) implies (I). In this case, if we scale all the vectors v_{α} by $C = (v_0 \hat{f}(0))^{-1} |P|$ then*

$$X_{[s]}(x) = \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) f(\gamma(x)) = Y_{[s]} F(x), \quad 0 \leq s < p.$$

Proof.

- a) Since f has accuracy p and translates of f along Γ are independent, by *Theorem 4.1* there exists a collection of row vectors $\{v_\alpha \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$ such that

$$X_{[s]}(x) = Y_{[s]}F(x) \quad 0 \leq s < p,$$

with $y_{[s]}$ and $Y_{[s]}$ given by (3) and (4) respectively.

Using the refinement equation $F(x) = LF(Ax)$ and the definition of $A_{[s]}$ we see that

$$Y_{[s]}F(Ax) = X_{[s]}(Ax) = A_{[s]}X_{[s]}(x) = A_{[s]}Y_{[s]}F(x) = A_{[s]}Y_{[s]}LF(Ax),$$

and since f has independent Γ -translates, this implies that $Y_{[s]} = A_{[s]}Y_{[s]}L$ for $0 \leq s < p$ which proves (ii).

To prove (i), consider the case $s = 0$. Since $y_{[0]}(\gamma) = v_0$ for all $\gamma \in \Gamma$ we have

$$1 = x^0 = X_{[0]}(x) = \sum_{\gamma \in \Gamma} v_0 f(\gamma(x)) \text{ a.e. and hence } v_0 \neq 0.$$

Further, if P is a fundamental domain of Γ then

$$v_0 \hat{f}(0) = v_0 \int_{\mathbb{R}^d} f(x) dx = v_0 \sum_{\gamma \in \Gamma} \int_P f(\gamma(x)) dx = \int_P 1 dx = |P| \neq 0,$$

which completes the proof of a).

- b) For each $0 \leq s < p$, define the vector-valued function $G_{[s]} : \mathbb{R}^d \rightarrow \mathbb{C}^{d_s}$, by

$$G_{[s]}(x) = \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) f(\gamma(x)) = Y_{[s]}F(x).$$

Note that for each fixed x , only finitely many terms in the sum defining $G_{[s]}(x)$ are nonzero.

Using the equation $Y_{[s]} = A_{[s]}Y_{[s]}L$ and the refinement equation $LF(Ax) = F(x)$, we have

$$(6) \quad G_{[s]}(Ax) = Y_{[s]}F(Ax) = A_{[s]}Y_{[s]}LF(Ax) = A_{[s]}Y_{[s]}F(x) = A_{[s]}G_{[s]}(x).$$

Since $X_{[s]}(Ax) = A_{[s]}X_{[s]}(x)$, we see that $G_{[s]}(x)$ and $X_{[s]}(x)$ behave identically under dilation by A . We will show that if we take $C = (v_0 \hat{f}(0)) |P|^{-1}$, then $G_{[s]}(x) = CX_{[s]}(x)$ for $0 \leq s < p$. So $G_{[s]}$ coincides with $X_{[s]}$, $0 \leq s < p$ - up to a constant that does not depend on s .

The quotient \mathbb{R}^d/\mathcal{L} is a compact abelian group, equipped with the normalized Haar measure. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathcal{L}$, be the canonical projection onto the quotient.

The map $\tau := \pi A \pi^{-1} : \mathbb{R}^d / \mathcal{L} \rightarrow \mathbb{R}^d / \mathcal{L}$ is well defined, preserves the measure and is a continuous and surjective endomorphism of the group $\mathbb{R}^d / \mathcal{L}$.

The group of the characters of $\mathbb{R}^d / \mathcal{L}$ is given by

$$(\mathbb{R}^d / \mathcal{L})^\wedge = \{\gamma_\lambda : \mathbb{R}^d / \mathcal{L} \rightarrow S; \gamma_\lambda(x) = e^{2\pi i \langle x, \lambda \rangle}, \text{ with } \lambda \in \mathcal{L}^*\}.$$

If $\gamma_\lambda \circ \tau^n = \gamma_\lambda$ for some $n \in \mathbb{N}$, then $e^{2\pi i \langle \tau^n x, \lambda \rangle} = e^{2\pi i \langle x, \lambda \rangle}$ for all $x \in \mathbb{R}^d / \mathcal{L}$ or equivalently $e^{2\pi i \langle x, (A^n)^t \lambda \rangle} = e^{2\pi i \langle x, \lambda \rangle}$ for all $x \in \mathbb{R}^d / \mathcal{L}$. Therefore $(A^n)^t \lambda = \lambda$ and since A^n is expansive, $\lambda = 0$. Hence $\gamma_\lambda \circ \tau^n = \gamma_\lambda$ if and only if $\gamma_\lambda = 1$. Therefore, by Theorem 1.10 of [15], the application τ is ergodic.

We now proceed by induction to show that $G_{[s]}(x) = C X_{[s]}(x)$ for $0 \leq s < p$ with C independent of s .

Consider the case $s = 0$, in which $G_{[0]}(x)$ is scalar-valued. Since $A_{[0]}$ is the constant 1, Eq. (6) states that $G_{[0]}(Ax) = G_{[0]}(x)$. Further, $y_{[0]}(\gamma) = v_0$ for every $\gamma \in \Gamma$, so $G_{[0]}(x) = \sum_{\gamma \in \Gamma} v_0 f(\gamma(x))$. Therefore, for each $\ell \in \mathcal{L}$ we have

$$G_{[0]}(x - \ell) = \sum_{\gamma \in \Gamma} v_0 f(\gamma(x - \ell)) = \sum_{\gamma \in \Gamma} v_0 f(\gamma \tau_\ell(x)) = \sum_{\gamma \in \Gamma} v_0 f(\gamma(x)).$$

Thus $G_{[0]}(x)$ satisfies

$$G_{[0]}(Ax) = G_{[0]}(x) \text{ and } G_{[0]}(x - \ell) = G_{[0]}(x) \text{ for all } \ell \in \mathcal{L}.$$

Hence $G_{[0]}(\tau(x)) = G_{[0]}(x)$ for each $x \in \mathbb{R}^d / \mathcal{L}$. Since τ is ergodic, it follows that $G_{[0]}$ is constant a.e on \mathfrak{L} , where \mathfrak{L} is the fundamental domain of \mathcal{L} (Theorem 1.6 of [15]). By periodicity, we therefore have $G_{[0]}(x) = C$ a.e. on \mathbb{R}^d . We can evaluate this constant explicitly, since

$$C|P| = \int_P G_{[0]}(x) dx = v_0 \sum_{\gamma \in \Gamma} \int_P f(\gamma(x)) = v_0 \int_{\mathbb{R}^d} f(x) dx = v_0 \hat{f}(0) \neq 0.$$

In particular $C = \left(v_0(\hat{f})(0) \right) |P|^{-1} \neq 0$. Suppose now, inductively, that $G_{[t]}(x) = C X_{[t]}(x)$ a.e. for $0 \leq t < s$. Then we have

$$\begin{aligned} G_{[s]}(x - \ell) &= Y_{[s]} F(x - \ell) = \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) f(\gamma \tau_{-\ell}(x)) = \sum_{\sigma \in \Gamma} y_{[s]}(\sigma \tau_\ell) f(\sigma(x)) \\ &= \sum_{t=0}^s Q_{[s,t]}(\ell) \sum_{\sigma \in \Gamma} y_{[t]}(\sigma) f(\sigma(x)) \text{ by Lemma 3.4.} \end{aligned}$$

This yields

$$\begin{aligned} G_{[s]}(x - \ell) &= \sum_{t=0}^s Q_{[s,t]}(\ell) Y_{[t]} F(x) = \sum_{t=0}^s Q_{[s,t]}(\ell) G_{[t]} \\ &= Q_{[s,s]}(\ell) G_{[s]}(x) + \sum_{t=0}^{s-1} Q_{[s,t]}(\ell) G_{[t]}(x). \end{aligned}$$

Using induction, we have

$$\begin{aligned} G_{[s]}(x - \ell) &= Q_{[s,s]}(\ell) G_{[s]}(x) + C \sum_{t=0}^{s-1} Q_{[s,t]}(\ell) X_{[t]}(x) \\ &= Q_{[s,s]}(\ell) G_{[s]}(x) + C \sum_{t=0}^s Q_{[s,t]}(\ell) X_{[t]}(x) - C Q_{[s,s]}(\ell) X_{[s]}(x) \\ &= G_{[s]}(x) + C X_{[s]}(x - \ell) - C X_{[s]}(x) \text{ by definition of } Q_{[s,t]}. \end{aligned}$$

Therefore, if we define $H_{[s]} = G_{[s]}(x) - C X_{[s]}(x)$ then

$$H_{[s]}(Ax) = A_{[s]} H_{[s]} \text{ and } H_{[s]}(x - \ell) = H_{[s]}(x), \text{ for } \ell \in L.$$

This implies that

$$H_{[s]}(\tau(x)) = A_{[s]} H_{[s]}(x).$$

Let now $E \subset \mathfrak{L}$ be a set of positive measure on which $H_{[s]}$ is bounded, say $\|H_{[s]}(x)\| \leq M$ for $x \in E$, where $\|\cdot\|$ is any fixed norm on \mathbb{C}^{d_s} . Since τ is ergodic, we know from Birkhoff's Ergodic Theorem (see [16]) that for almost every $x \in \mathfrak{L}$,

$$(7) \quad \lim_{n \rightarrow \infty} \frac{\#\{0 < k \leq n : \tau^k(x) \in E\}}{n} = |E| > 0.$$

Let $x \in P$ such that (7) holds. Then there exists an increasing sequence $\{n_j\}_{j=1}^\infty$ of positive integers such that $\tau^{n_j}(x) \in E$ for each j . Hence

$$M \geq \|H_{[s]}(\tau^{n_j}(x))\| = \|A_{[s]}^{n_j} H_{[s]}(x)\|.$$

Now, since A is expansive then $A_{[s]}$ is expansive, therefore $\|A_{[s]}^{n_j} H_{[s]}(x)\|$ diverges to infinity if $H_{[s]}(x) \neq 0$. Therefore we must have $H_{[s]}(x) = 0$ a.e. on P . Since $H_{[s]}$ is Λ -periodic, it must therefore vanish a.e. on \mathbb{R}^d . Hence $G_{[s]} = C X_{[s]}$ a.e., which completes the proof. \square

Since the conditions for accuracy given in the previous theorem are rather difficult to check, we follow [3] to give several equivalent formulations for condition (ii) in statement (II). The proof is very technical, so we will include it as an appendix.

Theorem 4.4. *Assume that $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is integrable, compactly supported and satisfies the refinement equation (1). Let $m = |\det A|$, and let $\gamma_1, \dots, \gamma_m$ be a full set of digits of the left cosets of Γ . Here, the left cosets Γ_i are $\Gamma_i = \gamma_i A \Gamma A^{-1}$.*

Given a collection $\{v_\alpha \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$ of row vectors, let

$$y_{[s]}(\gamma) = \sum_{t=0}^s Q_{[s,t]}(l) b_{[t]}^{-1} v_{[t]} \text{ and } Y_{[s]} = (y_{[s]}(\gamma))_{\gamma \in \Gamma}.$$

If $v_0 \neq 0$, then the following statements are equivalent:

- a) $Y_{[p-1]} = A_{[p-1]} Y_{[p-1]} L$. *Equivalently,*
 $y_{[p-1]}(\sigma) = A_{[p-1]} \sum_{\gamma \in \Gamma} y_{[p-1]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}}$ for $\sigma \in \Gamma$.
- b) $Y_{[s]} = A_{[s]} Y_{[s]} L$ for $0 \leq s < p$. *Equivalently,*
 $y_{[s]}(\sigma) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}}$ for $\sigma \in \Gamma$.
- c) $y_{[s]}(\gamma_i) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\gamma_i^{-1}}$ for $0 \leq s < p$ and $i = 1, \dots, m$.
- d) $v_{[s]} = \sum_{\gamma \in \Gamma_i} \sum_{t=0}^s Q_{[s,t]}(\gamma^{-1}) A_{[t]} v_{[t]} d_{\gamma^{-1}}$ for $0 \leq s < p$ and $i = 1, \dots, m$.

Note that by this theorem, if one wants to check for accuracy p , one does not need to check **all** conditions $0 \leq s < p$, but it is enough to check it for $s = p - 1$.

Note also, that in the particular case that the crystal group consists only of translations, the conditions to ensure the accuracy of a vector valued function given in [3, Theorem 4.8] coincide with the ones of the previous Theorem.

As in the translation case, for the single function case ($r = 1$) this theorem enables us to obtain a much nicer accuracy condition for f .

Theorem 4.5. *Let (Γ, G, Λ) be a splitting crystal triple, A a Γ -admissible matrix, $m = |\det(A)|$ and let $\Lambda_1, \dots, \Lambda_m$ be the (left) cosets of $\Lambda/A\Lambda$. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a Γ -refinable function. If the coefficients c_γ of the refinement equation (1) satisfy:*

- i) $\sum_{\gamma \in \Gamma} c_{\gamma^{-1}} = m$.
- ii) For each $b \in G$

$$(8) \quad \sum_{l \in \Lambda_1} l^\alpha c_{(b,l)^{-1}} = \dots = \sum_{l \in \Lambda_m} l^\alpha c_{(b,l)^{-1}} = \beta_{b,\alpha},$$

- iii) 1 is not eigenvalue of the matrix $\sum_{b \in G} \beta_{b,0} b_{[s]} A_{[s]}$ for each $0 \leq s < p$.

Then f has accuracy p .

These conditions should be compared to Theorem 3.7 in [3].

Proof. We define the matrices

$$M_{[s,t]} = \sum_{b \in G} \left[\sum_{l \in \Lambda_i} Q_{[s,t]}(l) c_{(b^{-1}, l)} \right] b_{[t]}.$$

By hypothesis the coefficients c_γ satisfy (8). Hence the sum

$$\sum_{l \in \Lambda_i} Q_{[s,t]}(l) c_{(b^{-1}, l)},$$

is independent of i . Moreover, as 1 not is eigenvalue of $M_{[s,s]} A_{[s]}$, $(I - M_{[s,s]} A_{[s]})$ is invertible. Therefore, if we define the vectors $v_{[s]}$ as follows

$$(9) \quad v_{[s]} = (I - M_{[s,s]} A_{[s]})^{-1} \sum_{t=0}^{s-1} M_{[s,t]} A_{[t]} v_{[t]},$$

we will show, that they satisfy condition **d)** of Theorem 4.4: For $0 \leq s < p$ and $i = 1, \dots, m$, we compute

$$\begin{aligned} & \sum_{\gamma \in \Gamma_i} \sum_{t=0}^s Q_{[s,t]}(\gamma^{-1}) A_{[t]} v_{[t]} c_{\gamma^{-1}} \\ &= \sum_{\gamma \in \Gamma_i} \sum_{t=0}^s Q_{[s,t]}(-b(l)) b_{[t]} A_{[t]} v_{[t]} c_{\gamma^{-1}} \quad (\text{by Definition 3.2}) \\ &= \sum_{b \in G} \sum_{\ell \in \Lambda_i} \sum_{t=0}^s Q_{[s,t]}(\ell) b_{[t]} A_{[t]} v_{[t]} c_{(b^{-1}, \ell)} \quad \text{where we define } \ell = -b(l) \\ &= \sum_{b \in G} \sum_{\ell \in \Lambda_i} b_{[s]} c_{(b^{-1}, \ell)} A_{[s]} v_{[s]} + \sum_{t=0}^{s-1} \left[\sum_{b \in G} \sum_{\ell \in \Lambda_i} Q_{[s,t]}(\ell) b_{[t]} c_{(b^{-1}, \ell)} \right] A_{[t]} v_{[t]} \\ &= M_{[s,s]} A_{[s,s]} v_{[s]} + \sum_{t=0}^{s-1} M_{[s,t]} A_{[t]} v_{[t]} = v_{[s]}. \end{aligned}$$

Therefore, f has accuracy p . □

4.3. Special vector functions. In this section we apply Theorem 4.5 to obtain accuracy conditions for a special case of vector (lattice)-refinable functions.

Given (Γ, G, Λ) a splitting crystal triple, with the point group $G = \{g_1, \dots, g_r\}$, in [14] the authors show that if we associate to a scalar function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ the vector valued function $F : \mathbb{R}^d \rightarrow \mathbb{C}^r$, $F = (f \circ g_1, \dots, f \circ g_r)$, then these two functions have properties in common.

The following definition is important for our purpose.

Definition 4.6. Let (Γ, G, Λ) be a splitting crystal triple and $G = \{g_1, g_2, \dots, g_r\}$. Let A be a Γ -admissible matrix and $\{c_k\}_{k \in \Lambda}$, with $c_k \in M_r(\mathbb{C})$. We will say that the matrices c_k have (Γ, A) -symmetry, if

$$c_{i,j}^k = c_{1,\rho_i(j)}^{g_{h_i}^{-1}(k)} \text{ for all } i, j = 1, \dots, r \text{ and } k \in \Lambda.$$

where h_i and ρ_i are permutations of $\{1, \dots, r\}$ such that

$$g_{h_i} = Ag_i A^{-1} \text{ and } g_{\rho_i(j)} = g_{h_i}^{-1} \circ g_j \text{ for each } i, j = 1, \dots, r.$$

In [14] it is shown that, under some (mild) conditions, f is Γ -refinable if and only if F is Λ -refinable. Precisely, they prove the following theorem.

Theorem 4.7. Let (Γ, G, Λ) be a splitting crystal triple, $G = \{g_1 = Id, \dots, g_r\}$, A a Γ -admissible matrix and $m = |\det A|$. We consider the sequence $\{c_\gamma\}_{\gamma \in \Gamma} \subset \mathbb{C}$ and $\{\tilde{c}_k\}_{k \in \Lambda} \subset M_r(\mathbb{C})$, where the matrices \tilde{c}_k are related to the scalars c_γ by the equality

$$\tilde{c}_k = (c_{i,j}^k)_{i,j=1,\dots,r} = \left(c_{(g_{h_i}^{-1} \circ g_j, g_j^{-1}(k))} \right)_{i,j=1,\dots,r}.$$

Then

- (1) If $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is Γ -refinable, then the function $F = (f, f \circ g_2^{-1}, \dots, f \circ g_r^{-1})$ is Λ -refinable and the coefficients of the Λ -refinement equation have (Γ, A) -symmetry.
- (2) If $\sum_{\gamma \in \Gamma} |c_\gamma|^2 < m$ and $F = (f_1, \dots, f_r) \in L^2(\mathbb{R}^d, \mathbb{C}^r)$ is the solution of the refinement equation associated to the matrices $\{\tilde{c}_k\}_{k \in \Lambda}$, then $F = (f \circ g_1, \dots, f \circ g_r)$ and the function $f = f_1$ is the solution of the Γ -refinement equation associated to the scalars $\{c_\gamma\}_{\gamma \in \Gamma}$, i.e., f is solution of

$$f(x) = \sum_{\gamma \in \Gamma} c_\gamma f(\gamma^{-1}ax) \quad \text{a.e. } x \in \mathbb{R}^d.$$

From Theorem 4.7 together with Theorem 4.5, we present a much simpler condition for characterizing the accuracy of some special functions $F : \mathbb{R}^d \rightarrow \mathbb{C}^r$.

Theorem 4.8. *Let (Γ, G, Λ) be a splitting crystal triple and $G = \{g_1, g_2, \dots, g_r\}$. Let A be a Γ -admissible matrix and $m = |\det A|$. Let $F : \mathbb{R}^d \rightarrow \mathbb{C}^r$ be a function such that $F = (f \circ g_1, \dots, f \circ g_r)$, is Λ -refinable and the coefficients \tilde{c}_k of the Λ -refinement equation have (Γ, A) -symmetry. Consider the scalars $c_\gamma = c_{(g_i, l)} = \tilde{c}_{1,i}^{g_i(l)} = (\tilde{c}_{g_i(l)})_{1,i}$, generated by the matrices \tilde{c}_k . If the sequence $\{c_\gamma\}_{\gamma \in \Gamma}$ satisfies the hypothesis of Theorem 4.5 and $\sum_{\gamma \in \Gamma} |c_\gamma|^2 < m$, then F has accuracy p .*

Compare this to the conditions of Theorem 3.4 in [3]. The conditions of the previous Theorem are clearly much easier to check!

Proof. Without loss of generality, we assume that $g_1 = Id$. By Theorem 4.7 $f = f_1$ is a Γ -refinable function, and $\{c_\gamma\}_{\gamma \in \Gamma}$ are the coefficients of the Γ -refinement equation. Further $\{c_\gamma\}_{\gamma \in \Gamma}$ satisfy the hypothesis of Theorem 4.5, therefore the function f has accuracy p .

To show that F has accuracy p let $P(x)$ a polynomial of degree less than p . Then

$$(10) \quad P(x) = \sum_{\gamma \in \Gamma} c_\gamma f(\gamma(x)) = \sum_{k \in \Lambda} \sum_{i=1}^n c_{(g_i, k)} f(g_i(x + k)) = \sum_{k \in \Lambda} C_k F(x + k).$$

Then F reproduces the same polynomials than f . Therefore F has accuracy p . \square

From equality (10) we have in fact the following result.

Corollary 4.9. *Let (Γ, G, Λ) be a splitting crystal triple and $G = \{g_1, g_2, \dots, g_r\}$. Let A be a Γ -admissible matrix and $m = |\det A|$. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$, $f \in L^2(\mathbb{R}^d)$ and $F : \mathbb{R}^d \rightarrow \mathbb{C}^r$ be defined by $F = (f \circ g_1, \dots, f \circ g_r)$. Then f has accuracy p if and only if F has accuracy p .*

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APPENDIX: THEOREM 4.4

Theorem 4.4 *Assume that $f : \mathbb{R}^d \rightarrow \mathbb{C}^r$ is integrable, compactly supported and satisfies the refinement equation (1). Let $m = |\det A|$, and let $\gamma_1, \dots, \gamma_m$ be a full set of digits of the left cosets of Γ . Here, the left cosets Γ_i are $\Gamma_i = \gamma_i A \Gamma A^{-1}$.*

Given a collection $\{v_\alpha \in \mathbb{C}^{1 \times r} : 0 \leq |\alpha| < p\}$ of row vectors, let

$$y_{[s]}(\gamma) = \sum_{t=0}^s Q_{[s,t]}(l) b_{[t]}^{-1} v_{[t]} \text{ and } Y_{[s]} = (y_{[s]}(\gamma))_{\gamma \in \Gamma}.$$

If $v_0 \neq 0$, then the following statements are equivalent:

- a) $Y_{[p-1]} = A_{[p-1]} Y_{[p-1]} L$. Equivalently,

$$y_{[p-1]}(\sigma) = A_{[p-1]} \sum_{\gamma \in \Gamma} y_{[p-1]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}} \text{ for } \sigma \in \Gamma.$$

- b) $Y_{[s]} = A_{[s]}Y_{[s]}L$ for $0 \leq s < p$. Equivalently,
 $y_{[s]}(\sigma) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}}$ for $\sigma \in \Gamma$.
- c) $y_{[s]}(\gamma_i) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\gamma_i^{-1}}$ for $0 \leq s < p$ and $i = 1, \dots, m$.
- d) $v_{[s]} = \sum_{\gamma \in \Gamma_i} \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma^{-1}) A_{[t]} v_{[t]} d_{\gamma^{-1}}$ for $0 \leq s < p$ and $i = 1, \dots, m$.

Proof. b) \Rightarrow a) and b) \Rightarrow c) are trivial. So we will prove a) \Rightarrow b), c) \Rightarrow b) and c) \Leftrightarrow d).

a) \Rightarrow b)

Assume that (a) holds, we consider for $j \in \mathcal{L}$, and $\sigma \in \Gamma$

$$(11) \quad \sum_{s=0}^{p-1} \mathcal{Q}_{[p-1,s]}(Aj) \left(A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}} \right).$$

Then by Lemma 3.1 we have that

$$\begin{aligned} (11) &= \sum_{s=0}^{p-1} A_{[p-1]} \mathcal{Q}_{[p-1,s]}(j) A_{[s]}^{-1} A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}} \\ &= A_{[p-1]} \sum_{\gamma \in \Gamma} \sum_{s=0}^{p-1} \mathcal{Q}_{[p-1,s]}(j) y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}} \\ &= A_{[p-1]} \sum_{\gamma \in \Gamma} y_{[p-1]}(\gamma \tau_j) d_{A\gamma A^{-1}\sigma^{-1}} \text{ by Lemma 3.4} \\ &= A_{[p-1]} \sum_{\gamma \in \Gamma} y_{[p-1]}(\gamma) d_{A\gamma A^{-1}(\sigma \tau_{Aj})^{-1}} \text{ where we take } \gamma' = \gamma \tau_j \\ &= y_{[p-1]}(\sigma \tau_{Aj}) = \sum_{s=0}^{p-1} \mathcal{Q}_{[p-1,s]}(Aj) y_{[s]}(\sigma) \text{ by Lemma 3.4} \end{aligned}$$

Then by Lemma 3.1 of [3] we have that

$$A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\sigma^{-1}} = y_{[s]}(\sigma),$$

for $0 \leq s < p$ and $\sigma \in \Gamma$, so statement (b) holds.

c) \Rightarrow b)

By hypothesis $y_{[s]}(\gamma_i) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\gamma_i^{-1}}$ for $0 \leq s < p$, $i = 1, \dots, m$ and each digit $\gamma_i = (b_i, l_i)$.

Let $\sigma \in \Gamma$ then there exists unique $i = 1, \dots, m$ and $\phi \in \Gamma$, such that $\sigma = \gamma_i A \phi A^{-1}$. Then $y_{[s]}(\sigma) = y_{[s]}(\gamma_i A \phi A^{-1})$ and by Lemma 3.4

$$(12) \quad y_{[s]}(\gamma_i A \phi A^{-1}) = \sum_{u=0}^s \mathcal{Q}_{[s,u]}(A \phi A^{-1}) y_{[u]}(\gamma_i) = \sum_{u=0}^s A_{[s]} \mathcal{Q}_{[s,u]}(\phi) A_{[u]}^{-1} y_{[u]}(\gamma_i)$$

By hypothesis

$$\begin{aligned} (12) &= \sum_{u=0}^s A_{[s]} \mathcal{Q}_{[s,u]}(\phi) A_{[u]}^{-1} A_{[u]} \sum_{\gamma \in \Gamma} y_{[u]}(\gamma) d_{A \gamma A^{-1} \gamma_i^{-1}} \\ &= \sum_{u=0}^s A_{[s]} \mathcal{Q}_{[s,u]}(\phi) \sum_{\gamma \in \Gamma} y_{[u]}(\gamma) d_{A \gamma A^{-1} \gamma_i^{-1}} \\ (13) \quad &= A_{[s]} \sum_{\gamma \in \Gamma} \left(\sum_{u=0}^s \mathcal{Q}_{[s,u]}(\phi) y_{[u]}(\gamma) \right) d_{A \gamma A^{-1} \gamma_i^{-1}}. \end{aligned}$$

Using Lemma 3.4 we have

$$\sum_{u=0}^s \mathcal{Q}_{[s,u]}(\phi) y_{[u]}(\gamma) = y_{[s]}(\gamma \phi).$$

Then

$$(13) = A_{[s]} \sum_{\gamma' \in \Gamma} y_{[s]}(\gamma') d_{A \gamma' A^{-1} \sigma^{-1}} \text{ where we again, set } \gamma' = \gamma \sigma.$$

Therefore

$$y_{[s]}(\sigma) = A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A \gamma A^{-1} \sigma^{-1}}.$$

$c) \Rightarrow d)$.

Assume that (c) holds, i.e. $y_{[s]}(\gamma_i) A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A \gamma A^{-1} \gamma_i^{-1}}$, for $0 \leq$

$s < p$ and $i = 1, \dots, m$. Then

$$\begin{aligned}
 v_{[s]} &= y_{[s]}(Id) = y_{[s]}(\gamma_i \gamma_i^{-1}) = \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_i^{-1}) y_{[t]}(\gamma_i) \\
 &= \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_i^{-1}) A_{[t]} \sum_{\gamma \in \Gamma} y_{[t]}(\gamma) d_{A\gamma A^{-1}\gamma_i^{-1}} \text{ by hypothesis} \\
 &= \sum_{\gamma \in \Gamma} \sum_{t=0}^s \sum_{u=0}^t \mathcal{Q}_{[s,t]}(\gamma_i^{-1}) A_{[t]} \mathcal{Q}_{[t,u]}(\gamma) v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
 &= \sum_{\gamma \in \Gamma} \sum_{t=0}^s \sum_{u=0}^t \mathcal{Q}_{[s,t]}(\gamma_i^{-1}) \mathcal{Q}_{[t,u]}(A\gamma A^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
 &= \sum_{\gamma \in \Gamma} \sum_{u=0}^s \sum_{t=u}^s \mathcal{Q}_{[s,t]}(\gamma_i^{-1}) \mathcal{Q}_{[t,u]}(A\gamma A^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
 &= \sum_{\gamma \in \Gamma} \sum_{u=0}^s \mathcal{Q}_{[s,u]}(A\gamma A^{-1}\gamma_i^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
 &= \sum_{\sigma \in \Gamma_i} \sum_{u=0}^s \mathcal{Q}_{[s,u]}(\sigma^{-1}) A_{[u]} v_{[u]} d_{\sigma^{-1}},
 \end{aligned}$$

where the last equality is obtained considering $\sigma \in \Gamma_i$, and then

$$\sigma = \gamma_i A \gamma^{-1} A^{-1}, \gamma^{-1} \in \Gamma.$$

$d) \Rightarrow c).$

Assume now that (d) holds. Then

$$\begin{aligned}
y_{[s]}(\gamma_i) &= \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_i) v_{[t]} \\
&= \sum_{t=0}^s \mathcal{Q}_{[s,t]}(\gamma_i) \sum_{\gamma \in \Gamma} \sum_{t=0}^u \mathcal{Q}_{[t,u]}(A\gamma A^{-1}\gamma_i^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
&= \sum_{\gamma \in \Gamma} \sum_{u=0}^s \sum_{t=u}^s \mathcal{Q}_{[s,t]}(\gamma_i) \mathcal{Q}_{[t,u]}(A\gamma A^{-1}\gamma_i^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
&= \sum_{\gamma \in \Gamma} \sum_{u=0}^s \mathcal{Q}_{[s,u]}(A\gamma A^{-1}\gamma_i^{-1}\gamma_i) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \text{ by Lemma 3.3 3,} \\
&= \sum_{\gamma \in \Gamma} \sum_{u=0}^s \mathcal{Q}_{[s,u]}(A\gamma A^{-1}) A_{[u]} v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \\
&= \sum_{\gamma \in \Gamma} \sum_{u=0}^s A_{[s]} \mathcal{Q}_{[s,u]}(\gamma) v_{[u]} d_{A\gamma A^{-1}\gamma_i^{-1}} \text{ by Lemma 3.3 4,} \\
&= \sum_{\gamma \in \Gamma} A_{[s]} \left(\sum_{u=0}^s \mathcal{Q}_{[s,u]}(\gamma) v_{[u]} \right) d_{A\gamma A^{-1}\gamma_i^{-1}} \\
&= A_{[s]} \sum_{\gamma \in \Gamma} y_{[s]}(\gamma) d_{A\gamma A^{-1}\gamma_i^{-1}}.
\end{aligned}$$

□

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